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Periodic Quasi-Exactly Solvable Models

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Various quasi-exact solvability conditions, involving the parameters of the periodic associated Lamé potential, are shown to emerge naturally in the quantum Hamilton–Jacobi (QHJ) approach. We study the singularity structure of the quantum momentum function, which yields the band-edge eigenvalues and eigenfunctions and compare it with the solvable and quasi-exactly solvable non-periodic potentials, as well as the periodic ones.

KEY WORDS: quasi-exactly solvable Hamiltonians; quantum Hamilton–Jacobi formalism; associated Lamé potential.

1. INTRODUCTION

Quasi-exactly solvable (QES) Hamiltonians, interconnecting a diverse array of physical problems, have been the subject of extensive study in recent times (Atre and Panigrahi, 2003; Geogo *et al.*, 2003; González-López *et al.*, 1991; Razavy, 1980, 1981; Shifman, 1989; Singh *et al.*, 1978; Turbiner and Ushveridze, 1987; Ushveridze, 1994; Znojil, 1983). These systems, containing a finite number of exactly obtainable eigenstates, have been linked with classical electrostatic problems, as also to the finite dimensional irreducible representations of certain algebras. Some of these studies employ group theoretical methods, others are based on the symmetry of the relevant differential equations (González-López *et al.*, 1991; Shifman, 1989; Turbiner and Ushveridze, 1987; Ushveridze, 1994). The key to the existence of the finite number of identifiable states is the quasi-exact solvability condition, relating certain potential parameters of these dynamical systems. An interesting feature, distinguishing these QES systems from known exactly solvable cases, is the presence of complex zeros in the polynomial part of the wave functions. Hence, quantum Hamilton–Jacobi (QHJ) formalism, being naturally formulated in the complex domain, is ideally suited for studying the

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QES problems (Geogo *et al.*, 2003). Although polynomial potentials have been studied rather exhaustively, QES periodic potentials have not received significant attention in the literature.

The associated Lamé potential (ALP),

$$V(x) = a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\,\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)} \tag{1}$$

is an interesting example of a periodic potential, which is exactly solvable, when $a = b$ and shows QES property, when $a \neq b$. Here, $\mathrm{sn}\,(x,m), \mathrm{cn}\,(x,m)$, and $\mathrm{dn}\,(x,m)$ are the doubly periodic elliptic functions with modulus parameter m (Hancock, 1958). The ALP has a periodic lattice of period $K(m)$ with the basis composed of two different atoms which are alternately placed. It possesses a surprisingly large variety of QES solvability conditions depending on the nature of the potential parameters a and b.

For a, b being unequal integers, with $a > b > 0$, there are a bound bands followed by a continuum band, whose band-edge solutions can be obtained analytically. If $a - b$ is odd (even) integer, it has b doubly degenerate band edges of period $2K(4K)$, which cannot be obtained analytically. The existence of analytically inaccessible states can be ascertained through the oscillation theorem. For a, b having half-integral values (with $a > b$), there are infinite number of bands with band-edge wave functions having period $2K$ $(4K)$, if $a - b$ is odd (even). Of these infinite number of bands, $a - b$ bands have band edges, which are non-degenerate with period $2K(4K)$ and $b + \frac{1}{2}$ doubly degenerate states of period $2K(4K)$ that can be obtained analytically. For a being an integer and b being a half-integer or vice versa, one can obtain some exact analytical results for mid-band states (Khare and Sukhatme, 1999, 2001; Magnus and Wrinkler, 1966).

It is quite natural to enquire about the origin of this rich QES structure in the associated Lamé potential (Ganguly, 2002; Tkachuk and Voznyak, 2002). This paper is devoted to the study of the same through the quantum Hamilton–Jacobi approach. The quasi-exact solvability conditions, involving the parameters of the periodic associated Lamé potential, are shown to emerge naturally. We also study the singularity structure of the quantum momentum function, which yields the band-edge eigenvalues and eigenfunctions. As will be seen in the text, the present approach is quite economical as compared to the earlier known methods, for determining the eigenvalues and eigenfunctions (Arscot, 1964; Magnus and Wrinkler, 1966).

In our earlier studies, we had looked at non-periodic ES, QES, and ES periodic potentials through the QHJ formalism which was initiated by Leacock and Padgett (1983). We were successful in obtaining the quasi-exact solvability condition (Geogo *et al.*, 2003) for QES models and in obtaining the eigenvalues and eigenfunctions for the ES and the band-edge eigenfunctions and eigenvalues of the ES periodic Lamé and the associated Lamé potentials (Sree Ranjani *et al.*, 2004). As noted earlier, when $a = b = j$, j being an integer, the associated Lamé potential is ES. The present study makes use of the QHJ formalism as developed in our earlier papers (Geogo *et al.*, 2003; Sree Ranjani *et al.*, 2004) for studying the quasi-exactly solvable regime of the associated Lamé potential. As has been seen in the non-periodic case, the QES potentials give rise to novel singularity structures in the QHJ approach, which is quite different from the ES cases. In this light, it is all the more interesting to study QES periodic potentials in the present approach. As will be seen in the text, new technical points did arise and had to be taken care of before the solutions could be found for these problems.

The QHJ formalism revolves around the logarithmic derivative of the wave function p,

$$p = -i\hbar \frac{d}{dx} \ln \psi, \tag{2}$$

known as the quantum momentum function (QMF), which satisfies a Riccati type equation

$$p^2 - i\hbar \frac{dp}{dx} = 2M(E - V(x)). \tag{3}$$

We have found that, the knowledge of the singularity structure of the QMF and the residue at these singular points are the only information, that is needed to obtain the QES condition and the solutions. In all our earlier studies, it was assumed that, *the point at infinity is an isolated singularity*. This assumption turned out to be correct and led to the QES conditions. The present study will enable us to analyze the singularity structure of the QMF, to check the assumptions and to see if one could get the QES conditions as a consequence of this assumption and reproduce known results of band-edge energies and eigenfunctions.

The plan of the paper is as follows. In Section 2, we obtain the QES condition along with the forms of the wave functions for the general ALP. In Section 3, we analyze the situation where both a, b are integers taking values 2 and 1, respectively. The case when a, b are both half-integers, with values $7/2$ and $1/2$, respectively, is analyzed in Section 4. Section 5 contains the concluding remarks.

2. QES CONDITION AND THE FORMS OF THE WAVE FUNCTIONS

The QHJ equation for the ALP, putting $\hbar = 2M = 1$, is given by

$$p^2 - ip' = \left(E - a(a+1)m\,\mathrm{sn}^2(x) - b(b+1)m\,\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)}\right). \tag{4}$$

Note that with the transformation $b \to -b - 1$ or $a \to -a - 1$, the potential does not change. Hence, for our analysis, without loss of generality, we take a, b

positive, with $a > b$. Defining $p \equiv -iq$ and substituting it in Eq. (4), one obtains

$$q^2 + \frac{dq}{dx} = a(a+1)m\,\mathrm{sn}^2(x) + b(b+1)m\,\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - E. \qquad (5)$$

Changing the variable to $t \equiv \mathrm{sn}\,(x)$, and writing

$$q = \sqrt{(1-t^2)(1-m\,t^2)}\phi, \quad \text{with} \quad \phi = \chi + \frac{1}{2}\left(\frac{m\,t}{1-m\,t^2} + \frac{t}{1-t^2}\right), \qquad (6)$$

Equation (5), gets transformed into

$$\chi^2 + \frac{d\chi}{dt} + \frac{m^2t^2 + 2m(1-2b(b+1))}{4(1-mt^2)^2} + \frac{2+t^2}{4(1-t^2)^2} + \frac{2E - mt^2(1-2a(a+1))}{2(1-t^2)(1-mt^2)} = 0. \qquad (7)$$

For all our later calculations, we shall treat χ as the QMF and Eq. (7) as the QHJ equation. It is interesting to note that through the above mapping the original eigenvalue problem has been cast onto the zero energy sector of a different Schrödinger problem as is evident in Eq. (7).

2.1. Singularity Structure

From Eq. (7), we see that χ has fixed poles at $t = \pm 1$ and $t = \pm\frac{1}{\sqrt{m}}$. In addition to the fixed poles, the QMF has finite number of moving poles and no other singular points in the complex plane. Hence, one can write χ as a sum of the singular and analytical parts as follows

$$\chi = \frac{b_1}{t-1} + \frac{b_1'}{t+1} + \frac{d_1}{t-\frac{1}{\sqrt{m}}} + \frac{d_1}{t+\frac{1}{\sqrt{m}}} + \frac{P_n'}{P_n} + Q(t)\,, \qquad (8)$$

where b_1, b_1', d_1, and d_1' are the residues at $t = \pm 1$ and $\pm\frac{1}{\sqrt{m}}$, respectively, which need to be calculated. P_n is an nth degree polynomial with $\frac{P_n'}{P_n} = \sum_{k=1}^{n}\frac{1}{t-t_k}$ being the summation of terms coming from the n moving poles with residue one. The function $Q(t)$ is analytic and bounded at infinity. Hence, from Liouville's theorem it is a constant, say C. The residues at the fixed poles can be calculated by taking the Laurent expansion around each individual pole and substituting them in Eq. (7). Comparing the coefficients of different powers of t, one gets two values of residues at each pole owing to the quadratic nature of the QHJ equation. Thus, the two values of the residues, b_1, b_1' at $t = \pm 1$ are

$$b_1 = \frac{3}{4}, \frac{1}{4} \quad \text{and} \quad b_1' = \frac{3}{4}, \frac{1}{4}\,. \qquad (9)$$

At $t = \pm\frac{1}{\sqrt{m}}$, one has

$$d_1 = \frac{3}{4} + \frac{b}{2}, \frac{1}{4} - \frac{b}{2} \quad \text{and} \quad d_1' = \frac{3}{4} + \frac{b}{2}, \frac{1}{4} - \frac{b}{2}. \tag{10}$$

Since, there is no way of ruling out one of the two values of these residues, we need to consider both the values. Imposing the parity constraint on the wave function yields $\chi(t) = -\chi(t)$, which in turn gives the condition $b_1 = b_1'$ and $d_1 = d_1'$.

2.2. Behavior at Infinity

Equation (8) gives the behavior of χ in the entire complex plane. Hence, for large t,

$$\chi \sim \frac{2b_1 + 2d_1 + n}{t}, \tag{11}$$

where the restriction $b_1 = b_1'$ and $d_1 = d_1'$ has been applied. This should match with the leading behavior of χ obtainable from Eq. (7). Note that, the assumption about the singularity structure of χ is equivalent to the point at infinity being an isolated singularity. Since χ has at most an isolated singular point at infinity, one can expand χ in Laurent series around the point at infinity as

$$\chi(t) = \lambda_0 + \frac{\lambda_1}{t} + \frac{\lambda_2}{t^2} + \cdots . \tag{12}$$

Substituting Eq. (12) in (7) and comparing various powers of t, one gets $\lambda_0 = 0$, thus making $Q(t)$ in Eq. (8) equal to zero. Further one obtains

$$\lambda_1 = a + 1, \quad -a. \tag{13}$$

Since both the equations, Eqs. (11) and (13), give the leading behavior of χ at infinity, both should be equal. Thus,

$$2b_1 + 2d_1 + n = \lambda_1. \tag{14}$$

Taking various combinations of b_1 and d_1 from Eqs. (9) and (10), substituting them in Eq. (14) one obtains the QES condition for each combination as given in Table I for $\lambda_1 = a + 1$. Thus, one sees that all the allowed combinations of residues give one of the forms of QES condition (Khare and Sukhatme, 1999), where $n = 0, 1, 2 \ldots$. Note that, the other value of λ_1, i.e., $-a$, when substituted instead of $a + 1$ in Eq. (14), gives the QES condition for negative values of a, b, i.e., for $a \to -a - 1,\ b \to -b - 1$ in $b_1,\ d_1$.

Table I. The Quasi-Exact Solvability Condition From the Four Permitted Combinations of b_1 and d_1 for the General Associated Lamé Potential

Set	b_1	d_1	$2b_1 + 2d_1 + n = \lambda_1$	QES condition
1	3/4	$\frac{3}{4} + \frac{b}{2}$	$2 + b + n = a$	$b - a = -n - 2$
2	3/4	$\frac{1}{4} - \frac{b}{2}$	$1 - b + n = a$	$a + b + 1 = n + 2$
3	1/4	$\frac{3}{4} + \frac{b}{2}$	$1 + b + n = a$	$b - a = -n - 1$
4	1/4	$\frac{1}{4} - \frac{b}{2}$	$-b + n = a$	$a + b = n$

2.3. Forms of Wave Function

From Eq. (2), one can write ψ in terms of p as

$$\psi(x) = \exp\left(\int i\, p dx\right). \tag{15}$$

Changing the variable to t and writing p in terms of χ, one gets

$$\psi(x) = \exp\left\{\int \left[\chi + \frac{1}{2}\left(\frac{mt}{1 - mt^2} + \frac{t}{1 - t^2}\right)\right] dt\right\}. \tag{16}$$

Substituting χ from Eq. (8) in the above equation gives the wave function in terms of the residue b_1, d_1 and the polynomial P_n

$$\psi(t) = \exp\left(\int \left[\frac{(1 - 4b_1)t}{2(1 - t^2)} + \frac{(1 - 4d_1)mt}{2(1 - mt^2)} + \frac{P_n'}{P_n}\right] dt\right). \tag{17}$$

In terms of the original variable x the wave function takes the form,

$$\psi(x) = (\mathrm{cn}\, x)^{\alpha} (\mathrm{dn}\, x)^{\beta} P_n(\mathrm{sn}\, x) \tag{18}$$

where $\alpha = \frac{4b_1 - 1}{2}$, $\beta = \frac{4d_1 - 1}{2}$. Hence, for each set of b_1, d_1 one gets a wave function given by Eq. (18). The degree n of this polynomial, which is obtained from Eq. (14)

$$n = a + 1 - 2b_1 - 2d_1 \tag{19}$$

in terms of either $a + b$ or $a - b$, as is evident from Table I. The forms of the wave functions can also be found, for the two different cases, when $a + b$ and $a - b$ are odd and even separately.

Case 1. *Both $a + b, a - b$ are even: We introduce $N = \frac{a+b}{2}$ and $M = \frac{a-b}{2}$, where M and N are integers, and obtain the forms of the wave functions in Table II, in terms of M and N, for the four sets of combinations of b_1 and d_1 in Table I.*

Case 2. *Both $a + b$, $a - b$ odd: Introducing $2N' + 1 = a + b$ and $2M' + 1 = a - b$, where M' and N' are integers, we obtain the wave functions, in terms of M' and N', for the four sets of combinations of b_1 and d_1 in Table III.*

Table II. The Form of the Wave Functions for the Four Sets of Residue Combinations When $a+b$ and $a-b$ are Even and Equal to $2N$ and $2M$, Respectively

Set	b_1	d_1	$n=\lambda_1-2b_1-2d_1$	$n(M,N)$	Wave function $\psi(x)$	LI solutions
1	3/4	$\frac{3}{4}+\frac{b}{2}$	$a-b-2$	$2M-2$	$\operatorname{cn}x(\operatorname{dn}x)^{1+b}P_{2M-2}(\operatorname{sn}x)$	M
2	3/4	$\frac{1}{4}-\frac{b}{2}$	$a+b-1$	$2N-1$	$\frac{\operatorname{cn}x}{(\operatorname{dn}x)^b}P_{2N-1}(\operatorname{sn}x)$	N
3	1/4	$\frac{3}{4}+\frac{b}{2}$	$a-b-1$	$2M-1$	$(\operatorname{dn}x)^{b+1}P_{2M-1}(\operatorname{sn}x)$	M
4	1/4	$\frac{1}{4}-\frac{b}{2}$	$a+b$	$2N$	$\frac{P_{2N}(\operatorname{sn}x)}{(\operatorname{dn}x)^b}$	$N+1$

From the forms of the wave functions in Tables II and III, one observes that the number of linearly independent solutions is different for the two cases. The unknown polynomial in the wave function can be obtained by substituting χ from Eq. (8) in the QHJ equation, which gives

$$P_n''(t)+4P_n(t)\left(\frac{b_1 t}{t^2-1}+\frac{md_1 t}{mt^2-1}\right)+G(t)P_n(t)=0 \tag{20}$$

where

$$G(t)=\frac{t^2\left(4b_1^2-2b_1+\frac{1}{4}\right)-2b_1+\frac{1}{2}}{(t^2-1)^2}+\frac{m^2t^2\left(4d_1^2-2d_1+\frac{1}{4}\right)-2md_1+m\left(\frac{1-2b(b+1)}{2}\right)}{(mt^2-1)^2}+\frac{2E+(16b_1d_1-1-2a(a+1))mt^2}{2(1-t^2)(1-mt^2)}.$$

The above differential equation is equivalent to a system of n linear equations for the coefficients of the different powers of t in $P_n(t)$. The energy eigenvalues are obtained by setting the corresponding determinant equal to zero. In the next section we obtain the band-edge wave functions for the associated Lamé potential, when $a=2$ and $b=1$.

Table III. The Form of the Wave Functions for the Four Sets of Residue Combinations When $a+b$ and $a-b$ are Odd and Equal to $2N'+1$ and $2M'+1$, Respectively

Set	b_1	d_1	$n=\lambda_1-2b_1-2d_1$	$n(M,N)$	Wave function $\psi(x)$	LI solutions
1	3/4	$\frac{3}{4}+\frac{b}{2}$	$a-b-2$	$2M'-1$	$\operatorname{cn}x(\operatorname{dn}x)^{1+b}P_{2M'-1}(\operatorname{sn}x)$	M'
2	3/4	$\frac{1}{4}-\frac{b}{2}$	$a+b-1$	$2N'$	$\frac{\operatorname{cn}x}{(\operatorname{dn}x)^b}P_{2N'}(\operatorname{sn}x)$	$N'+1$
3	1/4	$\frac{3}{4}+\frac{b}{2}$	$a-b-1$	$2M'$	$(\operatorname{dn}x)^{b+1}P_{2M'}(\operatorname{sn}x)$	$M'+1$
4	1/4	$\frac{1}{4}-\frac{b}{2}$	$a+b$	$2N'+1$	$\frac{P_{2N'+1}(\operatorname{sn}x)}{(\operatorname{dn}x)^b}$	$N'+1$

3. ALP WITH *a*, *b* INTEGERS

For this case, we consider the associated Lamé potential with $a = 2,\ b = 1$. For the purpose of comparison with literature, we work with the supersymmetric potential

$$V_{-}(x) = 6m\,\mathrm{sn}\,(x)^2 + 2m\frac{\mathrm{cn}\,x^2}{\mathrm{dn}\,x^2} - 4m \tag{21}$$

This potential is the same as Eq. (1) with $a = 2$ and $b = 1$, except that a constant has been added to make the lowest energy equal to zero. Note that for this potential the combination $a + b$ and $a - b$ are both odd, i.e., 3 and 1, respectively. Hence, we use Table III to obtain all the information regarding the residues at the fixed poles, number of moving poles of χ, number of linearly independent solutions and their form, for each set, by taking the values of $a = 2$, $b = 1$, $M' = 0$, and $N' = 1$. The unknown polynomial in the wave function can be obtained from Eq. (20), where $G(t)$ for this potential satisfies,

$$G(t) = \frac{t^2\left(4b_1^2 - 2b_1 + \frac{1}{4}\right) - 2b_1 + \frac{1}{2}}{(t^2-1)^2} + \frac{m^2t^2\left(4d_1^2 - 2d_1 + \frac{1}{4}\right) - 2md_1 - \frac{3m}{2}}{(mt^2-1)^2} + \frac{2E + 8m + (16b_1d_1 - 13)mt^2}{2(1-t^2)(1-mt^2)} \tag{22}$$

Using Eqs. (20) and (22), one gets the explicit expressions for the eigenfunctions and the eigenvalues as given in Table IV. From the table, we see that the first set of residues gives $n = -1$, which will not be considered as n cannot be negative. Thus, this particular case of Lamé potential has five band-edge solutions, which can be obtained analytically out of an infinite number of possible states.

Table IV. For the Associated Lamé Potential $V_{-}(x) = 6m\,\mathrm{sn}\,(x)^2 + 2m\frac{\mathrm{cn}\,(x)^2}{\mathrm{dn}\,(x)^2} - 4m$, With $a = 2$ and $b = 1$ the Residues, the Value of n, Number of Linear Independent Solutions, the Band-Edge Eigenfunctions and Eigenvalues are as Follows. Here $a + b = 3$ and $a - b = 1$ That Gives $N' = 1$ and $M' = 0$

Set	b_1	d_1	n	LI solutions	Eigenfunction $\psi(x)$	Eigenvalues
1	$3/4$	$5/4$	-1	—	—	—
2	$3/4$	$-1/4$	2	2	$\frac{\mathrm{cn}\,x}{\mathrm{dn}\,x}(3m\mathrm{sn}^2x - 2 \pm\sqrt{4-3m})$	$5 - 3m \pm 2\sqrt{4-3m}$
3	$1/4$	$5/4$	0	1	dn^2x	0
4	$1/4$	$-1/4$	2	2	$\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x}(3m\mathrm{sn}^2x - 2 - m \pm\sqrt{4-5m+m^2})$	$5 - 2m \pm 2\sqrt{m^2-5m+4}$

Table V. For the Associated Lamé Potential $V_{-}(x) = \frac{63}{4} m \,\mathrm{sn}^2 x + \frac{3}{4} m \frac{\mathrm{cn}^2 x}{\mathrm{dn}^2 x} - 2 - \frac{29}{4} m + \delta_9$ With $a = 7/2$ and $b = 1/2$ the Residues, the Value of n, Number of Linear Independent Solutions, the Band-Edge Eigenfunctions and Eigenvalues are as Follows. Here $a + b = 4$ and $a - b = 3$ That Gives $N = 2$ and $M' = 1$

Set	b_1	d_1	n	LI solutions	Eigenfunction $\psi(x)$	Eigenvalues
1	3/4	1	1	1	$\mathrm{cn}\, x(\mathrm{dn}\, x)^{3/2}\mathrm{sn}\, x$	$\delta_9 - m + 2$
2	3/4	0	3	2	$\mathrm{cn}\, x(\mathrm{dn}\, x)^{3/2}\mathrm{sn}\, x$	$\delta_9 - m + 2$
					$\mathrm{cn}\, x(\mathrm{dn}\, x)^{-1/2}\mathrm{sn}\, x(1 - 2\mathrm{sn}^2 x)$	$14 - 7m + \delta_9$
3	1/4	1	2	2	$(\mathrm{dn}\, x)^{3/2}(12m\,\mathrm{sn}^2 x - 5m - 2 - \delta_9)$	0
					$(\mathrm{dn}\, x)^{3/2}(12m\,\mathrm{sn}^2 x - 5m - 2 + \delta_9)$	$2\delta_9$
4	1/4	0	4	3	$(\mathrm{dn}\, x)^{3/2}(12m\,\mathrm{sn}^2 x - 5m - 2 - \delta_9)$	0
					$(\mathrm{dn}\, x)^{3/2}(12m\,\mathrm{sn}^2 x - 5m - 2 + \delta_9)$	$2\delta_9$
					$1 - 8\mathrm{sn}^2 x\mathrm{cn}^2 x$	$14 - 7m + \delta_9$

4. ALP WITH *a*, *b* HALF-INTEGERS

The potential studied here is the supersymmetric associated Lamé potential, with $a = 7/2$ $b = 1/2$:

$$V_{-} = \frac{63}{4} m \,\mathrm{sn}^2 x + \frac{3}{4} m \frac{\mathrm{cn}^2 x}{\mathrm{dn}^2 x} - 2 - \frac{29}{4} m + \delta_9 \tag{23}$$

where $\delta_9 = \sqrt{4 - 4m + 25m^2}$. Note that, for this case, $a + b = 4$ and $a - b = 3$, these are even and odd, respectively. Hence, for such cases, one needs to use sets 1 and 3 from Table II and sets 2 and 4 from Table III in order to get the four groups of the eigenfunctions. The solutions for this potential are given in Table V. We see that there is a degeneracy in the band-edge energy eigenvalue, $14 - 7m + \delta_9$.

5. CONCLUSIONS

In this study, we have demonstrated the applicability of QHJ formalism to QES periodic potentials. We have been successful in obtaining the quasi-exact solvability conditions and band-edge solutions for cases when both a and b are integers or half-integers. The origin of the large number of solvability conditions, for the ALP case, comes out naturally in QHJ approach. One sees that there are four groups of solutions with each group having different number of moving poles. We would like to point out that in the case of non-periodic QES models, the parameter n picks out a particular QES model within a family of potentials. All the analytically solvable solutions of this potential will have n moving poles, whereas in the periodic case, by fixing n and one of the parameter a, one obtains two potentials. For example, if one chooses $n = 4$ and $a = 7/2$, and substitutes them in the four QES conditions in Table I, one obtains the values of b as $-5/2, 3/2, -3/2$, and $1/2$. Among these the values $-5/2, 3/2$ and $-3/2, 1/2$ correspond to the same potential owing to the transformation $b = -b - 1$. Thus, one has two potentials corresponding to distinct (a, b) values namely, $(7/2, 3/2)$ and $(7/2, 1/2)$. Both these potentials will have a group of solutions which have a QMF with four moving poles.

Interestingly, the singularity structures of the QES and the ES periodic potentials are similar. But a comparison with polynomial potentials shows that the singularity structure of the periodic potentials is different.

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